Hand-in assignments in Advanced Quantum Mechanics, spring semester 2021.

These are hand in assignments for the course in Advanced Quantum mechanics at the Masaryk University in the spring of year 2019. They are the first part of the requirement of the course, the second being an oral exam. The problems should be handed in minimum one week before the oral exam. **Do not leave out any part of the calculations and motivate your assumptions and approximations carefully**. You my answer in Czech *or* English. The required minimum number of points is **25** evenly distributed over the different topics.

The formalism

1. Imagine that you are given a nonorthonormal basis $|i\rangle$ so that $B_{ik} = \langle i|k\rangle$ is a general invertible matrix. Find an expression for the unity operator in this basis. Write an arbitrary state $|\psi\rangle$ in this basis, *i.e.* find an expression for c_k in

$$|\psi\rangle = \sum_k |k\rangle\, c_k$$

Also, if A is any operator, find an expression for the representation of this operator in the given basis

$$A = \sum_{k,l} |k\rangle\, A_{kl}\, \langle l|$$

(4p)

Propagators and Path Integrals

1. A charged particle moving in a one dimensional space with an electric field F has a Hamiltonian given by

$$\hat{H} = \frac{\hat{p}^2}{2m} - F\hat{x}\ .$$

Derive the propagator for this system in the following way: Since the Hamiltonian is independent of time, argue that you may write the time evolution operator as

$$\hat{U}(t,t') = e^{-\frac{i}{\hbar}(t-t')\hat{H}}$$

Use the Baker-Campbell-Hausdorff formula to write the time evolution operator as

$$\hat{U}(t,t') = e^{f(\hat{x})}e^{g(\hat{p})}$$

You may now calculate the configuration space propagator $\langle x|\,\hat{U}(t,t')\,|x'\rangle$ by cleverly inserting the unit operator. Evaluate the resulting integral to find the final expression. **(6p)**

2. Calculate the propagator for a particle in a linear potential

$$S[x(t)] = \int dt(\frac{1}{2}m\dot{x}^2 - Fx)$$

using path integral methods. Here are some useful observations that you might want to use

a) In the path integral, we sum over all paths *with the prescribed boundary conditions.*

b) The sum will be the same if we shift all paths by some particular *fixed* path.

c) Define the new path $y(t)$ as the old path shifted by a solution of the equations of motion $x_{cl}(t)$ so that $y = x - x_{cl}$.

d) However, shifting a path satisfying a particluar boundary condition by a fixed path gives a new path that usually does not satisfy the same boundary condition. What boundary conditions should $y(t)$ fulfil if the classical solution x_{cl} satisfies the same boundary conditions as x?

e) Find a particular x_{cl} with the same boundary conditions as x, *i.e.* that begins at x' at time t' and ends at x at time t.

f) Show that the action $S[y(t)]$ consists of only of a kinetic term and a term dependent only on the boundary conditions. In particular there is no potential for $y(t)$.

g) The path integral over $y(t)$ can now be done using the result for the path integral of a free particle. Remeber that is is given by

$$\int \mathcal{D}x e^{\frac{i}{\hbar}S_{free}[x(t)]} = \sqrt{\frac{m}{2\pi i\hbar(t-t')}}e^{\frac{im(x-x')^2}{2\hbar(t-t')}}$$

for a path that starts at x' at time t' and ends at x at time t.

Check that your result agrees with the result of the previous problem. **(6p)**

Relativistic quantum mechanics

1. If we interpret $\rho = \psi^\dagger\psi$ as the particle density, use the Dirac Hamiltonian to define a current $\mathbf{j}$ such that the equation of continuity is fulfilled

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0$$

   In order to do this it is useful to have the relation

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$$

   which you should prove. Calculate ρ and $\mathbf{j}$ for the plane wave $\psi(\mathbf{x}) = u(\mathbf{p}) e^{-\frac{i}{\hbar}Et} e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}$ normalized so that $\psi^\dagger \gamma^0 \psi = 1$. Comment on your result. **(4p)**

2. Prove the following trace identities valid for any representation

$$\begin{aligned} \mathrm{tr}(\gamma^\mu) &= 0 \\ \mathrm{tr}(\gamma^\mu\gamma^\nu) &= 4\eta^{\mu\nu} \\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma) &= 0 \\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) \end{aligned}$$

   using *only* the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ and the properties of the trace. **(4p)**

3. In a two-dimensional space-time with coordinates (t, x) the Klein-Gordon equation looks like

$$\frac{1}{c^2}\partial_t^2 \phi - \partial_x^2 \phi + \left(\frac{mc}{\hbar}\right)^2 \phi = 0$$

   derive the Dirac equation by using the fact that $(E - pc)(E + pc) = E^2 - p^2c^2$. What do the gamma matrices look like? Show that they satisfy the defining relations of a Clifford algebra. **(4p)**

4. Study the Dirac equation in two dimensions using the gamma matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

How does the Dirac Hamiltonian look like? By making an ansatz for a plane wave solution $\psi = u(E,p)e^{-\frac{i}{\hbar}Et+\frac{i}{\hbar}px}$ find the solutions of the Dirac Hamiltonian Schrödinger equation for $u(E,p)$ for both negative and positive energy and normalized so that $u^\dagger u = \frac{|E|}{mc^2}$. What is the relation between E, p and m that needs to be fulfilled to have a solution?
Introduce a constant potential by adding a constant term V to the Dirac Hamiltonian and again make the the plane wave ansatz to find what the relation between E, p and m (and also V) is in order for a solution to exist. If we shoot particles with a constant energy $E > mc^2$ on a potential step of size V, the particles will of course penetrate into the potential region for V being very small but if we continue to raise the value of V the particles will stop being able to go into the potential region. At what value of the potential V does this happen? Imagine continuing to raise the potential V, is there any change in the behavior of the incoming electrons at some even higher value of V? **(4p)**

Scattering theory

1. Imagine studying particle scattering in a two dimensional world (there are plenty of examples of effectively two dimensional physical systems in Condensed Matter Physics). How would one define the cross-section? What dimension (unit) would it have? (**2p**)

2. Show how the wavefunction in the 2D scattering problem must look like far away from the source of scattering. Express the differential cross section in terms of the general solution of the wave function. (**4p**)

3. Determine, using the first Born approximation in the two dimensional case defined above, the differential and the total scattering cross-section in the low energy limit for a spherical potential well

$$V = \begin{cases} |V_0| & \text{for} \quad r < a \\ 0 & \text{for} \quad r > a. \end{cases}$$

   (**5p**)

4. Develop the partial wave method for the two dimensional scattering problem discussed above. First solve the free Schrödinger equation to find out the proper basis function that should be used in the two

dimensional case. Using these basis functions, write the plane wave $\frac{1}{(2\pi)}e^{ikx}$ in polar coordinates. (**5p**)

5. Using the partial wave method applied to the two dimensional problem show that the cross section for scattering on a hard sphere of radius R (well it should be called a hard disc in the two dimensional case) is given by the formula

$$\sigma = \frac{4}{k}\sum_{n=-\infty}^{\infty}\frac{J_n^2(kR)}{J_n^2(kR)+N_n^2(kR)}$$

where $k=\sqrt{2mE}/\hbar$. What is the total cross section in the low energy limit? (**5p**)

Useful formulas that may be used without further proof.

$$\begin{aligned}\int\frac{d^2l}{(2\pi)^2}\frac{e^{i\bar{l}\cdot\bar{x}}}{E-\frac{\hbar^2\bar{l}^2}{2m}+i\epsilon} &= -\frac{m}{\pi\hbar^2}K_0\left(-ik|\bar{x}|\right)\ , \quad E=\frac{\hbar^2k^2}{2m}\\ \lim_{|z|\to\infty}K_0(z) &= \sqrt{\frac{\pi}{2z}}e^{-z}\end{aligned}$$

where K_0 is a so called modified Bessel function.
The Bessel functions J_n and N_n are the two linearly independent solutions to the differential equation:

$$\frac{d^2\psi}{dz^2}+\frac{1}{z}\frac{d\psi}{dz}+\left(1-\frac{n^2}{z^2}\right)\psi=0$$

Here are some asymptotic formulas for ordinary Bessel functions:

$$\begin{aligned}\lim_{z\to 0}J_n(z) &= \frac{1}{n!}\left(\frac{z}{2}\right)^n\\ \lim_{z\to 0}N_0(z) &= \frac{2}{\pi}\ln\left(\frac{z}{2}\right)\\ \lim_{z\to 0}N_n(z) &= -\frac{(n-1)!}{\pi}\left(\frac{2}{z}\right)^n\\ \lim_{z\to\infty}J_n(z) &= \sqrt{\frac{2}{\pi z}}\cos\left(z-(2n+1)\frac{\pi}{4}\right)\\ \lim_{z\to\infty}N_n(z) &= \sqrt{\frac{2}{\pi z}}\sin\left(z-(2n+1)\frac{\pi}{4}\right)\end{aligned}$$

and some other useful relations

$$
\begin{aligned}
J_{-n}(z) &= (-1)^n J_n(z) \\
N_{-n}(z) &= (-1)^n N_n(z) \\
\int_0^{\pi} d\theta e^{ia\cos\theta} &= \pi J_0(a) \\
\int_0^1 dx x J_0(ax) &= \frac{1}{a} J_1(a)
\end{aligned}
$$